under consideration.

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INVESTIGATION OF THE OSCILLATIONS OF ESSENTIALLY NON-LINEAR SYSTEMS WITH INTERNAL RESONANCE*

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Oscillations in systems which do not become linear when the small parameter becomes equal to zero are studied. It is assumed that the generating system contains odd-order resonances. Conditionally periodic solutions of the generating and complete systems are constructed with an accuracy of up to first order in the small parameter. The results obtained represent a further development of the theory of bifurcation of the growth of a cycle from a position of equilibrium.

1. Let us consider an essentially non-linear quasi-autonomous system of $2n$-th order differential equations

$$u_k^{\cdot} = i\nu_k u_k + A_k v^p/v_k + \sum_{l\geqslant 1} \mu^l U_{kl}(u, v, t) \tag{1.1}$$
$$v_k^{\cdot} = \bar{u}_k^{\cdot};\quad v_k = \bar{u}_k,\quad v^p = v_1^{p_1} v_2^{p_2} \ldots v_n^{p_n},\quad A_k = \text{const}$$

where μ is a small parameter. The functions U_{kl} are polynomials in u_k, v_k $(k = 1, \ldots, n)$ of an arbitrarily large degree, vanishing when $u = v = 0$, with coefficients conditionally t-periodic and represented by a generalized finite Fourier series. The series in the parameter μ are absolutely convergent when its values are sufficiently small, and the point $u = v = 0$ is a unique singularity in the domain of variation of u and v in question.

We assume that the frequencies are connected by an odd-order resonance relation

$$p_1\nu_1 + \ldots + p_n\nu_n = 0$$
$$(p_i > 0\ (i = 1, \ldots, n),\ p = \Sigma p_l = 2m + 1\ (m = 1, 2, \ldots))$$

We note that when we have the internal odd-order resonance and no resonance relations of the same order connecting the eigenfrequencies with the frequencies of the conditionally periodic coefficients, we can reduce, to system (1.1), the arbitrary system of equations of perturbed motion with n pairs of the purely imaginary roots of the form

$$x_k^{\cdot} = -\nu_k y_k + X_k^{(p-1)} + X_k^{(p)} + \ldots,\quad y_k^{\cdot} = \nu_k x_k + Y_k^{(p-1)} + Y_k^{(p)} + \ldots$$

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where $X_k^{(m)}(x, y, t)$, $Y_k^{(m)}(x, y, t)$ are the m-th order forms in x and y, which can have periodic, as well as conditionally periodic coefficients.

Indeed, passing to the complex conjugate variable $u_k = x_k + iy_k$, $v_k = x_k - iy_k$ and carrying out the necessary transformations given in /1-3/, we arrive at the system

$$u_k^{\cdot} = i\nu_k u_k + A_k v^p/v_k + U_k^{(p)}(u, v, t) + \ldots, \quad v_k^{\cdot} = \bar{u}_k^{\cdot}$$

Introducing into this system a small parameter by means of the substitution

$$u_k = \mu w_k e^{i\alpha\nu_k t}, \quad \bar{u}_k = v_k = \mu \bar{w}_k e^{-i\alpha\nu_k t} \quad (\alpha = 1 - \mu^{p-2})$$

changing the time scale thus $\tau = \mu^{p-2}t$ and restoring the variables $w_k, \bar{w}_k, \tau$ to the previous notation u, v, t, we obtain a system of the form (1.1).

We shall consider the degenerate case, when

$$D_{1i} = a_1 b_i - b_1 a_i = 0 \quad (i = 1, \ldots, n) \qquad (1.2)$$
$$a_i = \operatorname{Re} A_i, \quad b_i = \operatorname{Im} A_i$$

We pose the problem of determining the stationary solution, in the sense of /1/, of system (1.1) in terms of the first order in μ, which become, when $\mu = 0$, the conditionally periodic solutions of the truncated system

$$u_k^{\cdot} = i\nu_k u_k + A_k v^p/v_k, \quad v_k^{\cdot} = \bar{u}_k^{\cdot} \qquad (1.3)$$

We note that condition (1.2) can be made to hold also in the case when all values of D_{1i} are of the order of μ^{l_i} $(l_i \geqslant 1)$, whereas A_i are not small.

As we known /2, 3/, (1.2) represents the necessary condition of stability of the zeroth solution of system (1.1) only when $n = 2$. However, the case when (1.2) holds is of considerable interest since it happens, in particular, in the case of Hamiltonian systems. When (1.2) holds, the zeroth solution of system (1.3) is stable if and only if the sequence of numbers $b_1, \ldots, b_n$ $(a_1, \ldots, a_n)$ contains at least one change of sign. Let us write $z_i = -\operatorname{sign} b_i$ and pass to real variables with the help of the substitution

$$u_k = (|b_k/b_1| r_k)^{1/2} e^{i\theta_k} \quad (k = 1, \ldots, n) \qquad (1.4)$$

We assume that

$$b_1 > 0, \; b_i < 0 \; (i = 2, \ldots, n_1), \; b_j > 0 \; (j = n_1 + 1, \ldots, n)$$

and omit, for brevity, the case when the numbers b_k contain zero values, since the corresponding equations are then quasilinear and can be included in the discussion in Sect.2.

As a result of the substitution (1.4), system (1.3) will take the form

$$r_k^{\cdot} = -2z_k D r^{p/2} \cos\gamma, \qquad (1.5)$$
$$\theta_k^{\cdot} = \nu_k + z_k D r^{p/2} r_k^{-1} \sin\gamma$$

where

$$\gamma = p_1\theta_1 + \ldots + p_n\theta_n - \varphi_a \qquad (1.6)$$
$$\varphi_a = \arg A_k \pm {}^1/_2 (z_k + 1)\pi \quad (k = 1, \ldots, n)$$
$$D = |A_1| \prod_{i=2}^{n} \left|\frac{b_i}{b_1}\right|^{p_i/2}, \quad r^{p/2} = r_1^{p_1/2} \ldots r_n^{p_n/2}$$

System (1.5) admits of n first integrals

$$r_1 + r_2 = R \qquad (1.7)$$
$$r_1 + r_i = R(1 + \tau_i) \quad (i = 3, \ldots, n_1; \; \tau_i > -1)$$
$$r_1 - r_j = -R\tau_j \quad (j = n_1 + 1, \ldots, n; \; \tau_j > -1)$$
$$r^{p/2} \sin\gamma = h \quad (R > 0, \; \tau_i, \tau_j, h - \text{const}) \qquad (1.8)$$

and can be reduced, by a simple change of variables, to the Hamiltonian form which is obtained in normalized form.

The condition that all r_i are non-negative yields the domain of definition of r_1: $R_1 < r_1 < R_2$, where

$$R_1 = \max(0, -R\tau_{n_1+1}, \ldots, -R\tau_n)$$
$$R_2 = \min(R, R((1 + \tau_3), \ldots, R(1 + \tau_{n_1}))$$

Further discussions and arguments carried out in Sect.1 are part of the process of integrating system (1.5).

Taking into account expressions (1.7), we can show that the equation

$$S(r_1) = \sum_{i=1}^{n} \frac{z_i p_i}{r_i} = 0 \qquad (1.9)$$

has only a single solution $r_{10} = R\alpha_1$, in the interval (R_1, R_2), and $0 < \alpha_1 < 1$.

Let us introduce the variable x such, that

$$r_i = R(\alpha_i - z_i x) \ (i = 1, \ldots, n) \tag{1.10}$$

This is clearly possible, provided that

$$\alpha_2 = 1 - \alpha_1, \ \alpha_i = \alpha_2 + \tau_i \ (i = 3, \ldots, n_1), \ \alpha_j = \alpha_1 + \tau_j \ (j = n_1 + 1, \ldots, n) \tag{1.11}$$

The variable x lies within the limits $\beta_1 < x < \beta_2$ where $\beta_i = -\alpha_1 + R_i/R$. When $x = \beta_i$, at least one of the numbers r_k will become zero.

Let us introduce the notation

$$y = -r^{p/2} \cos \gamma \tag{1.12}$$

From (1.8) and (1.12) we obtain the relation

$$h^2 + y^2 = r^p \tag{1.13}$$

Using (1.10), we shall write r^p as a polynomial in x:

$$r^p = R^p \left(k_0 + k_1 x + k_2 x^2 - \frac{1}{2} \alpha^p \sum (p_i/\alpha_i^2) H(x)\right) \tag{1.14}$$

where $H(x)$ is a polynomial of degree p, beginning with the third-order terms. It can be confirmed that

$$k_0 = \alpha^p, \quad k_1 = 0, \quad k_2 = -\frac{1}{2} \alpha^p \sum (p_i/\alpha_i^2)$$

We can show that $r^p/R^p \leqslant \alpha^p$ for any $r_1 \in (R_1, R_2)$, i.e. $x^2 + H(x) \geqslant 0$ for any $x \in (\beta_1, \beta_2)$. From the definition of α_1 and (1.10) it follows that all $\alpha_i > 0$.

Let us write

$$z = [{}^1/_2 \alpha^p \Sigma (p_i/\alpha_i^2)]^{1/2} x [1 + H(x)/x^2]^{1/2} = d_1 x + d_2 x^2 + d_3 x^3 + \ldots \tag{1.15}$$

Then, assuming that $d_1 \neq 0$, we obtain

$$x = \frac{1}{d_1} z - \frac{d_2}{d_1^3} z^2 + \frac{2d_2^2 - d_1 d_3}{d_1^5} z^3 + \ldots \tag{1.16}$$

We shall assume that x and z are sufficiently small for series (1.15) and (1.16) to converge.

From (1.13) and (1.15) we have

$$y^2/R^p + z^2 = \rho^2 \tag{1.17}$$

$$\rho^2 = \alpha^p - h^2/R^p \tag{1.18}$$

Introducing the variable φ with help of the formulas

$$z = \rho \cos \varphi, \ y = R^{p/2} \rho \sin \varphi \tag{1.19}$$

we shall form a system of equations in θ_1 and φ. To do this, we obtain from (1.5), (1.10) and (1.13)

$$x^{\cdot} = -2DR^{-1} y, \ y^{\cdot} = 2DR^{p-1} z \, dz/dx \tag{1.20}$$

and find

$$\varphi^{\cdot} = k \, dz/dx, \ k = 2DR^{p/2-1} \tag{1.21}$$

It can be shown that $\varphi^{\cdot} > 0$ when $x \in (\beta_1, \beta_2)$.

From (1.15), (1.8), (1.10) and (1.21) we obtain

$$\frac{d\theta_1}{d\varphi} = \left(\nu_1 - \frac{Dh}{R(x + \alpha_1)}\right) \frac{1}{k} \frac{dx}{dz}$$

Expanding its right-hand side in a Maclaurin series in z, making use of (1.19) and then integrating, we obtain a relation connecting θ_1 and φ:

$$\theta_1 = \left(\left(\nu_1 - \frac{Dh}{R\alpha_1}\right) \frac{1}{kd_1} + a_0^{(1)}\right) \varphi + \sum_{l \geqslant 1} a_l^{(1)} \sin l\varphi + C_1 \tag{1.22}$$

$$a_0^{(1)} = \frac{1}{\pi} \sum_{j \geqslant 1} g_j^{(1)} \rho^j \int_0^\pi \cos^j \varphi \, d\varphi$$

$$a_l^{(1)} = \frac{2}{\pi l} \sum_{j \geqslant l} g_j^{(1)} \rho^j \int_0^\pi \cos^j \varphi \cos l\varphi \, d\varphi, \quad g_j^{(1)} = g_j^{(1)}(h, R, \tau_m)$$

where $a_l^{(1)} = a_l^{(1)}(\rho, h, R, \tau_m)$ are Fourier coefficients.

Analogous expressions can be obtained for the remaining θ_l.

Thus we have obtained solutions of system (1.3)

$$u_l = (|b_l/b_1| r_l)^{1/2} e^{i\theta_l}, \quad r_l = R(\alpha_l - z_l x) \tag{1.23}$$
$$\theta_l = \nu_l^0 \varphi + \sum_{j\geqslant 1} a_j^{(l)} \sin j\varphi + C_l$$
$$x = \frac{1}{d_1} z - \frac{d_2}{d_1^3} z^2 + \frac{2d_2^2 - d_1 d_3}{d_1^5} z^3 + \ldots, \qquad z = \rho\cos\varphi$$
$$\left(\nu_l^0 = \left(\nu_l + \frac{z_l Dh}{R\alpha_l}\right)\frac{1}{kd_1} + a_0^{(l)}\right)$$

The relation $\varphi = \varphi(t)$ can be found in the same way as (1.22), by integrating Eq.(1.21)

$$t = \frac{1}{k_0}\varphi + \sum_{l\geqslant 1} \psi_l \sin l\varphi + \psi_0 \tag{1.24}$$

The solution (1.23), (1.24) contains the constants R, $\tau_3, \ldots, \tau_n$, h, ψ_0, $C_1, \ldots, C_n$, ρ, of which two (e.g. the last two) can be expressed in terms of the remaining constants.

We can assume that $R, h, \tau_3, \ldots, \tau_n, \theta_1, \ldots, \theta_{n-1}, \varphi$ are the new variables. Then the trajectories of the system will lie on n-dimensional tori.

The solution (1.23), (1.24) is periodic in the case when all numbers ν_l^0 $(l = 1, \ldots, n-1)$ are rational, and conditionally periodic otherwise. The rational character of ν_n^0 need not be checked, since the expansion for θ_n in (1.23) is dependent. From (1.6) it follows that

$$\theta_n = p_n^{-1}(\gamma - p_1\theta_1 - \ldots - p_{n-1}\theta_{n-1} + \varphi_a)$$

and γ can be uniquely expressed in terms of 2π-periodic function φ from the formula (1.8), (1.12), (1.10), (1.16) and (1.19).

2. We shall seek the conditionally periodic solutions of system (1.1) in terms of first order in μ, which become, when $\mu = 0$, the solution (1.23), (1.24) corresponding to the constants R_0, $\tau_{30}, \ldots, \tau_{n0}$, h_0, $C_{10}, \ldots, C_{n-1,0}$, ψ_{00}.

Let us find the derivatives of the integrals of the truncated system h, R $\tau_3, \ldots, \tau_n$, by virtue of the complete system

$$r_k^{\cdot} = -2z_k D r^{p/2}\cos\gamma + \sum_{l\geqslant 1} \mu^l R_{kl}(r, \theta, t) \tag{2.1}$$

$$\theta_k^{\cdot} = \nu_k + z_k D\frac{r^{p/2}}{r_k}\sin\gamma + \sum_{l\geqslant 1} \frac{\mu}{r_k} T_{kl}(r, \theta, t)$$

obtained from (1.1) using the substitution (1.4). Here R_{kl} and T_{kl} have the form (the summation is carried out over $m_i \geqslant 0$, $m \geqslant 1$, $|l_i| \leqslant m_i$).

$$\Sigma r^{m/2}(a^{(m,l)}(t)\cos(l_1\theta_1 + \ldots + l_n\theta_n) + b^{(m,l)}(t)\sin(l_1\theta_1 + \ldots + l_n\theta_n)) \tag{2.2}$$
$$(m = m_1 + \ldots + m_n)$$

Differentiating (1.7) and (1.8) we obtain, by virtue of (2.1),

$$R^{\cdot} = \sum_{l\geqslant 1} \mu^l (R_{1l} + R_{2l}) \tag{2.3}$$
$$R\tau_i^{\cdot} = \sum_{l\geqslant 1} \mu^l (R_{1l} + R_{il}) - R^{\cdot}(1 + \tau_i) \quad (i = 3, \ldots, n_1)$$
$$R\tau_j^{\cdot} = \sum_{l\geqslant 1} \mu^l (R_{jl} - R_{1l}) - R^{\cdot}\tau_j \quad (j = n_1 + 1, \ldots, n)$$
$$h^{\cdot} = h\Sigma\frac{p_i}{2r_i}\sum_{l\geqslant 1}\mu^l R_{il} - y\sum\frac{p_i}{r_i}\sum_{l\geqslant 1}\mu^l T_{il}$$

From (1.19) it follows that

$$\varphi^{\cdot} = \rho^{-2}R^{-p}[y^{\cdot} z R^{p/2} - (z^{\cdot} R^{p/2} + {}^1/_2 pzR^{p/2-1}R^{\cdot})y]$$

Carrying out the necessary algebra, we obtain

$$\varphi^{\cdot} = k\frac{\partial z}{\partial x} + \sum_{l\geqslant 1}\mu^l \Phi_l(r, \theta, y, z, t) \tag{2.4}$$

where $\partial z/\partial x$ is the partial derivative of the right-hand side of relation (1.15) and the functions Φ_l have the form (2.2) with the coefficients $a^{(m,l)}$, $b^{(m,l)}$ analytic in y, z and depending also on $R, \tau_3, \ldots, \tau_n, h$.

In order to reduce the amount of calculation, we shall introduce the vectors $I = (h, R, \tau_3, \ldots, \tau_n)$ and $C = (C_1, \ldots, C_{n-1}, \psi_0)$.

We shall find the dependence of θ_l and t on φ, from the expressions

$$\theta_l = \nu_l^0(I_0)\varphi + \sum_{j\geqslant 1} a_j^{(l)}(I_0)\sin j\varphi + C_l$$
$$t = \frac{1}{k_0(I_0)}\varphi + \sum_{j\geqslant 1}\psi_j(I_0)\sin j\varphi + \psi_0$$

where C_l, ψ_0 are the variables and I_0 denote certain unperturbed values of I which will be found later.

Differentiating the last expressions, we obtain

$$\frac{dC_l}{d\varphi} = \nu_l^0(I) + \sum_{j\geqslant 1} ja_j^{(l)}(I)\cos j\varphi + \sum_{m\geqslant 1}\mu^m F_{lm}(r, \theta, t, I, \varphi) - \tag{2.5}$$
$$\nu_l^0(I_0) - \sum_{j\geqslant 1} ja_j^{(l)}(I_0)\cos j\varphi \quad (l = 1, \ldots, n-1)$$
$$\frac{d\psi_0}{d\varphi} = \frac{1}{k_0(I)} + \sum_{j\geqslant 1} j\psi_j(I)\cos j\varphi + \sum_{m\geqslant 1}\mu^m \Psi_{0m}(r, \theta, t, I, \varphi) -$$
$$\frac{1}{k_0(I_0)} - \sum_{j\geqslant 1} j\psi_j(I_0)\cos j\varphi$$

Let us pass in (2.3) to the derivatives in φ, and rewrite (2.3) and (2.5) as follows:

$$\frac{dI}{d\varphi} = \sum_{l\geqslant 1}\mu^l H_l(r, \theta, t, I, \varphi) \tag{2.6}$$
$$\frac{dC}{d\varphi} = f(I, \varphi) - f(I_0, \varphi) + \sum_{l\geqslant 1}\mu^l S_l(r, \theta, t, I, \varphi)$$

where the functions H_l, S_l have the form (2.2) with coefficients $a^{(m,l)}$, $b^{(m,l)}$ depending on I, φ.

Let us replace in H_l, S_l the variables r, θ, t by their expressions in terms of I, C, φ. As a result we obtain

$$\frac{dI}{d\varphi} = \sum_{l\geqslant 1}\mu^l J_l(I, C, \varphi) \tag{2.7}$$
$$\frac{dC}{d\varphi} = f(I, \varphi) - f(I_0, \varphi) + \sum_{l\geqslant 1}\mu^l Z_l(I, C, \varphi)$$

We can show, as in the case $n = 2$*,(*Korolev I.A. On the oscillations of essentially non-linear systems with resonance. Moscow, Paper deposited in VINITI, 5.8.85, 5824-85. 1985.) that the functions J_l, Z_l are analytic in I, C in some neighbourhood of the unperturbed values of I_0, C_0 $(h_0 \neq 0,\ R_0 \neq 0)$, and conditionally periodic in φ, and the case when not a single number $p_1, \ldots, p_n$ is equal to unity is more cumbersome when it comes to practical calculations. We can also show that for fixed I, C, the values of the functions J_l, Z_l averaged over φ in the interval $(0, \infty)$, are independent of C when there are no resonances between the frequencies of the conditionally periodic coefficients on the right-hand sides of (2.7).

Using the conditionally periodic change of variables

$$I' = I - \mu u(I, C, \varphi),\ C' = C - \mu v(I, C, \varphi)$$

we will reduce (2.7) to the form

$$\frac{dI'}{d\varphi} = \mu B_1(I') - \mu\frac{\partial u}{\partial C}(f(I', \varphi) - f(I_0, \varphi)) + \cdots \tag{2.8}$$
$$\frac{dC'}{d\varphi} = f(I', \varphi) - f(I_0, \varphi) + \mu G_1(I') - \mu\frac{\partial v}{\partial C}(f(I', \varphi) - f(I_0, \varphi)) + \cdots$$
$$B_1(I') = \lim_{T\to\infty}\frac{1}{T}\int_0^T J_1(I', C', \varphi)\,d\varphi$$
$$u(I, C, \varphi) = \int(J_1(I, C, \varphi) - B_1(I))\,d\varphi$$

$$G_1(I') = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(Z_1(I', C', \varphi) + \frac{\partial f(I', \varphi)}{\partial I'} u(I', C', \varphi) \right) d\varphi$$

$$v(I, C, \varphi) = \int \left(Z_1(I, C, \varphi) + \frac{\partial f}{\partial I} u(I, C, \varphi) - G_1(I) \right) d\varphi$$

(the integrals are taken at fixed I, C, and repeated dots terms with μ, of degree higher than the first).

Let us choose n constants $I_0 = (h_0, R_0, \tau_{30}, \ldots, \tau_{n0})$ from the equations

$$B_1(I_0) = 0 \tag{2.9}$$

introduce the perturbations ξ: $I' = I_0 + \xi$, and write the equations of perturbed motion

$$\frac{d\xi}{d\varphi} = \mu A\xi + \mu M\xi + \mu A^{(2)} + \ldots, M = \frac{\partial u(I_0, C, \varphi)}{\partial C} \frac{\partial f}{\partial I} \tag{2.10}$$

where the mean value of the matrix M is zero, $A^{(2)}(\xi)$ is a set of terms beginning from the second order in ξ, repeated dots denote terms beginning with the second order in μ.

We note that when there are no resonances between the frequencies of the conditionally periodic coefficients on the right-hand sides of (2.7), the functions B_1 and G_1 in (2.8) may contain linear combinations of the components of the vector C. In this case we must supplement Eqs.(2.9) with equations equating the corresponding linear combinations of the components of the vector function G_1, to zero.

We obtain conditionally periodic solutions of system (1.1) stationary in the sense of /1/, with an accuracy of up to first order in μ, by substituting $I' = I_0$, $C' = C_0 + \mu G_1(I_0)\varphi$ into the transformations which reduce (2.8) to (1.1). Let all eigenvalues of the matrix A of (2.10) have negative real roots. Then* (*see /4/ and: Seregin V.N. On the study of the oscillations of systems with almost periodic coefficients. Candidate Dissertation, Moscow, MAI, 1980.) the corresponding stationary solution will be stable, and for sufficiently small μ it will differ arbitrarily little from the solution of the complete system. By the stability and nearness we mean the stability and nearness of the corresponding deformed tori. When $n = 2$, the conditionally periodic solutions will themselves be orbitally stable. This can be explained by the fact that in this case the motion can be described in terms of the variables R, h, θ_1, γ, the variable θ_1 can be replaced by φ, and the behaviour of γ will be governed by the behaviour of R, h, φ (this follows from (1.8), (1.12), (1.18) and (1.19)).

Thus we have described a method of constructing stable, conditionally periodic solutions of system (1.1) with an accuracy up to terms of first order in μ, differing as little as we choose from the corresponding solutions of the complete system becoming, when $\mu = 0$, solutions (1.23) and (1.24) of the truncated system.

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